VC dimension and Eluder dimension

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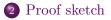
JULY 22, 2020

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2 Proof sketch

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DEFINITION 4. An action *a* is *VC-independent* of $\tilde{A} \subseteq A$ if for any $f, \tilde{f} \in \mathcal{F}$, there exists some $\tilde{f} \in \mathcal{F}$, which agrees with f on a and with \tilde{f} on \tilde{A} ; that is, $\tilde{f}(a) = f(a)$ and $\tilde{f}(\tilde{a}) = \tilde{f}(\tilde{a})$ for all $\tilde{a} \in \tilde{A}$. Otherwise, a is *VC-dependent* on \tilde{A} .

DEFINITION 5. The VC dimension of a class of binary-valued functions with domain \mathcal{A} is the largest cardinality of a set $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ such that every $a \in \tilde{\mathcal{A}}$ is VC-independent of $\tilde{\mathcal{A}} \setminus \{a\}$.

- Only one function in ℱ.
 In this case, VC dimension = 1. But applying the new definition we have VC dimension = ∞.
- Only two functions in ℱ.
 VC dimension = 1 or 2. But applying the new definition we have VC dimension = ∞.

Theorem

If $0, 1 \in \mathscr{F}$, then the two definitions are equal.

Proof.

It's trivial that the old definition implies the new definition. We focus on the other direction.

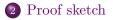
 $\forall f\in 2^{\mathscr{A}},$ where $|\mathscr{A}|=\mathrm{dim}VC.$ Let $X=\{a:f(a)=1\},$ by the definition, we have

$$\underset{a\in X}{\cup}\{g:g(Y)=1,g(\mathscr{A}\backslash Y)=0,Y=Y\cup\{a\}\}\subset\mathscr{F}$$

whereas $f \in \bigcup_{a \in X} \{g : g(Y) = 1, g(\mathscr{A} \setminus Y) = 0, Y = Y \cup \{a\}\} \subset \mathscr{F}$, so $f \in \mathscr{F}$.

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• The T-proi
od regret of a policy π is defined by

Regret
$$(T, \pi, \theta)$$
 $\sum_{t=1}^{T} \mathbb{E}[\max_{a \in \mathscr{A}_t} f_{\theta}(a) - f_{\theta}(A_t)|\theta].$

• The T-period Bayesian regret is defined by $\mathbb{E}[\operatorname{Regret}(T, \pi, \theta)]$, where the expectation is taken with respect to the prior distribution over θ . Hence,

BayesRegret
$$(T, \pi) = \sum_{t=1}^{T} \mathbb{E}[\max_{a \in \mathscr{A}_t} f_{\theta}(a) - f_{\theta}(A_t)].$$

- Assumption 1. For all $f \in \mathscr{F}$ and $a \in \mathscr{A}$, $f(a) \in [0, C]$.
- Assumption 2. For all $t \in \mathbb{N}$, $R_t f\theta(A_t)$ conditioned on (H_t, θ, A_t) is σ -sub-Gaussian.

Let $A_t^* = \max_{a \in \mathscr{A}_t} f_{\theta}(a)$, we have

BayesRegret
$$(T, \pi) = \sum_{t=1}^{T} \mathbb{E}[f_{\theta}(A_t^*) - f_{\theta}(A_t)].$$

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Step 1: **Regret decomposition.** If $f_{\theta}(A_t) \in [L_t(A_t), U_t(A_t)]$, then

$$\mathbb{E}[f_{\theta}(A_t^*) - f_{\theta}(A_t)] \le \mathbb{E}[U_t(A_t^*) - f_{\theta}(A_t)] \le \mathbb{E}[U_t(A_t) - f_{\theta}(A_t)].$$

Else

$$f_{\theta}(A_t^*) - f_{\theta}(A_t) \le C.$$

 $\Rightarrow \text{BayesRegret} \leq \mathbb{E} \sum_{t=1}^{T} [(U_t(A_t) - f_\theta(A_t)] + C \sum_{t=1}^{T} \mathbb{P}(f_\theta(A_t^*)) > U_t(A_t^*)).$

Step 2: Build confidence set. Minimize MSE loss $\rightarrow f_{\theta}(A_t) \in [L_t(A_t), U_t(A_t)]$ with high probability.

Step 3: Bound the confidence set interval. We have $U_t(A_t) - f_{\theta}(A_t) \leq \omega_t(A_t) := U_t(A_t) - L_t(A_t)$, and bound $\sum_{t=1}^{T} \omega_t(A_t)$ to conclude the proof.

Step 1:

PROPOSITION 1. For any UCB sequence $\{U_t \mid t \in \mathbb{N}\}$,

BayesRegret
$$(T, \pi^{PS}) = \mathbb{E} \sum_{t=1}^{T} [U_t(A_t) - f_{\theta}(A_t)] + \mathbb{E} \sum_{t=1}^{T} [f_{\theta}(A_t^*) - U_t(A_t^*)]$$

for all $T \in \mathbb{N}$.

$$L_{2,t}(f) = \sum_{1}^{t-1} (f(A_t) - R_t)^2$$
$$||\bar{f} - \underline{f}||_{2,E_t}^2 = \sum_{k=1}^{t-1} (\bar{f} - \underline{f})^2 (A_k)$$

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Step 2:

Lemma 3. For any $\delta > 0$ and $f: \mathcal{A} \mapsto \mathbb{R}$, with probability at least $1 - \delta$, $L_{2, t}(f) \ge L_{2, t}(f_{\theta}) + \frac{1}{2} \|f - f_{\theta}\|_{2, E_{t}}^{2} - 4\sigma^{2} \log(1/\delta)$

simultaneously for all $t \in \mathbb{N}$.

$$\beta_t^*(\mathcal{F}, \delta, \alpha) := 8\sigma^2 \log(N(\mathcal{F}, \alpha, \|\cdot\|_{\infty})/\delta) + 2\alpha t (8C + \sqrt{8\sigma^2 \ln(4t^2/\delta)}).$$

PROPOSITION 6. For all $\delta > 0$ and $\alpha > 0$, if

$$\mathcal{F}_{t} = \{ f \in \mathcal{F} \colon \| f - \hat{f}_{t}^{LS} \|_{2, E_{t}} \leq \sqrt{\beta_{t}^{*}(\mathcal{F}, \delta, \alpha)} \}$$

for all $t \in \mathbb{N}$, then

$$\mathbb{P}\left(f_{\theta} \in \bigcap_{t=1}^{\infty} \mathcal{F}_{t}\right) \geq 1 - 2\delta.$$

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Step 3:

LEMMA 4. For all $T \in \mathbb{N}$, if $\inf_{\rho \in \mathcal{F}_{\tau}} f_{\rho}(a) \leq f_{\theta}(a) \leq \sup_{\rho \in \mathcal{F}_{\tau}} f_{\rho}(a)$ for all $\tau \in \mathbb{N}$ and $a \in \mathcal{A}$ with probability at least 1 - 1/T, then

BayesRegret
$$(T, \pi^{PS}) \leq C + \mathbb{E} \sum_{t=1}^{T} w_{\mathcal{F}_t}(A_t).$$

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Step 3:

PROPOSITION 8. If $(\beta_t \ge 0 \mid t \in \mathbb{N})$ is a nondecreasing sequence and $\mathcal{F}_t := \{f \in \mathcal{F} : \|f - \hat{f}_t^{LS}\|_{2,E_t} \le \sqrt{\beta_t}\}$, then

$$\sum_{t=1}^{T} \mathbf{1}(w_{\mathcal{F}_t}(A_t) > \boldsymbol{\epsilon}) \leq \left(\frac{4\beta_T}{\boldsymbol{\epsilon}^2} + 1\right) \dim_E(\mathcal{F}, \boldsymbol{\epsilon})$$

for all $T \in \mathbb{N}$ and $\epsilon > 0$.

LEMMA 5. If $(\beta_t \ge 0 \mid t \in \mathbb{N})$ is a nondecreasing sequence and $\mathcal{F}_t := \{f \in \mathcal{F} : \|f - \hat{f}_t^{LS}\|_{2, E_t} \le \sqrt{\beta_t}\}$, then $\sum_{t=1}^T w_{\mathcal{F}_t}(A_t) \le 1 + \dim_E(\mathcal{F}, T^{-1})C + 4\sqrt{\dim_E(\mathcal{F}, T^{-1})\beta_T T}$

for all $T \in \mathbb{N}$.

Step 3:

DEFINITION 1. The Kolmogorov dimension of a function class \mathcal{F} is given by

$$\dim_{K}(\mathcal{F}) = \limsup_{\alpha \downarrow 0} \frac{\log N(\mathcal{F}, \alpha, \|\cdot\|_{\infty})}{\log(1/\alpha)}.$$

In particular, we have the following result.

PROPOSITION 7. For any fixed class of functions \mathcal{F} ,

 $\beta_t^*(\mathcal{F}, 1/t^2, 1/t^2) = 16(1 + o(1) + \dim_K(\mathcal{F}))\log t.$

PROPOSITION 10. For any fixed class of functions \mathcal{F} ,

 $\operatorname{BayesRegret}(T, \pi^{PS}) \le 1 + [\dim_{E}(\mathcal{F}, T^{-1}) + 1]C + 16\sigma\sqrt{\dim_{E}(\mathcal{F}, T^{-1})(1 + o(1) + \dim_{K}(\mathcal{F}))\log(T)T}]$

Thanks!

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