Finite-Time Error Bounds For Linear Stochastic Approximation and TD Learning

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Main Statement

Derive the finite error bounds on the moments of the error of the linear stochastic approximation algorithm:

\[ \Theta_{k+1} = \Theta_k + \epsilon (A(X_k) \Theta_k + b(X_k)) \] (1)

1. \( \{X_k, k \geq 0\} \) is an underlying Markov chain
2. \( A(X_k) \) is a random matrix; \( b(X_k) \) is a random vector; \( \Theta_k \) is a random vector
3. algorithm updates \( \Theta_k \) using recursion (1)
4. \( \epsilon \) is a constant step size
Outline

1. Motivation: $TD(0)$
2. Linear Stochastic Approximation
3. Finite-Time Error Bounds
TD Learning: TD(0)

Setup:
1. MDP over a finite space $S = \{1, \ldots, N\}$
2. Fix a stationary policy $\mu$
3. $\{Z_k\}$: the resulting Markov chain
4. Value function

$$V(i) := \mathbb{E} \left[ \sum_{k=0}^{\infty} \alpha^k c(Z_k, \mu(Z_k), Z_{k+1}) \middle| Z_0 = i \right]$$

(2)

where $c$ is one-step reward.

5. Purpose: estimate the value function $V$ associated with $\mu$ by observing a trajectory $\{z_0, z_1, z_2, \ldots\}$
TD(0): Linear Approximation

1. $V$ satisfies the Bellman equation: $V = T_\mu V$

$$V(i) = \mathbb{E}_j[c(i, \mu(i), j) + \alpha V(j)] = \mathbb{E}[c(i, \mu(i), j)] + \alpha \sum_j p_{ij} V(j)$$

(3)

denote $\bar{c} := (\mathbb{E}[c(1, \mu(i), j)], \ldots, \mathbb{E}[c(N, \mu(i), j)])^t$

2. If the transition probabilities $p_{ij}$ are known, we can solve (3) to get $V$.

3. still, when $N = |S|$ is large, we approximate value function $V$ by a linear function of feature functions $\phi^t(i) = (\phi_1(i), \ldots \phi_d(i))$:

$$V(i) \approx \sum_{k=1}^d \theta_k \phi_k(i)$$

(4)

where $d$ is small compared to $N$. Now: estimate weights $\theta_k$
1. Goal: approximate $V$ by a member from $\mathcal{L} = \{\phi^t \theta : \theta \in \mathbb{R}^d\}$

2. Minimizing $L^2$-error

$$\theta^* = \arg \min_{\theta \in \mathbb{R}^d} \| V - \phi^t \theta \|_\xi^2$$  

(5)

where

$$\| f \|_\xi^2 := \int_S f^2(s) \xi(ds)$$  

(6)

3. $\Pi_{\mathcal{L}} :=$ projection operator onto $\mathcal{L}$ with respect to $\| \|_\xi^2$; Solve the projected Bellman equation:

$$\Pi_{\mathcal{L}} T_\mu(\phi^t \theta) = \phi^t \theta$$  

(7)

4. since $\theta^*$ should satisfy

$$T_\mu(\phi^t \theta^*) \approx V$$  

(8)
TD Learning: Algorithm Design

1. one can show $\prod_{L} T_{\mu}$ is a contraction mapping when $\xi$ is chosen to be the stationary distribution of $\{Z_k\}$

2. by solving (7), one can show it is equivalent to solving for $\theta^*$ so that

$$
\mathbb{E}[\phi(i)(\phi(i)^t \theta^* - \alpha \phi(j)^t \theta^* - c(i, \mu(i), j)] = 0 \quad (9)
$$

3. observe that

$$
\theta^* - \epsilon \mathbb{E}[\phi(i)(\phi(i)^t \theta^* - \alpha \phi(j)^t \theta^* - c(i, \mu(i), j)] = \theta^* \quad (10)
$$

4. for an episode $\{Z_0, Z_1, \ldots\}$,

$$
\Theta_{k+1} = \Theta_k - \epsilon \phi(Z_k) (\phi^t(Z_k) \Theta_k - c(Z_k) - \alpha \phi^t(Z_{k+1}) \Theta_k) \quad (11)
$$

where $\Theta_k$ is the estimate of $\theta^*$ at time $k$, $\epsilon \in (0, 1)$ is a constant
TD(0): Convergence

Theorem (Tsitsiklis, Van Roy 1997)

$\Theta_k$ converges to $\theta^*$ where

$$\Pi_L T_\mu(\phi^t \theta^*) = \phi^t \theta^*$$  \hspace{1cm} (12)

Srikant and Ying 2019 provides finite-time error bounds on $\mathbb{E}\|\Theta_k - \theta^*\|^2$. Rewrite (11) as

$$\Theta_{k+1} = \Theta_k + \epsilon(A(X_k)\Theta_k + b(X_k))$$  \hspace{1cm} (13)

where

$$X_k := (Z_k, Z_{k+1}), \quad A(X_k) := -\phi(Z_k)(\phi^t(Z_k) - \alpha\phi^t(Z_{k+1}))$$  \hspace{1cm} (14)

and

$$b(X_k) := c(Z_k)\phi(Z_k) - A(X_k)\theta^*, \quad \Theta \leftarrow \Theta - \theta^*$$  \hspace{1cm} (15)
Assumptions

From now on, we focus on linear stochastic recursion (1). We use 2-norm for all vectors and induced 2-norm for all matrices.

Assumptions:

1. \( \{X_k\} \) is a Markov chain with state space \( S \).

\[
\lim_{k \to \infty} E[A(X_k)] = \bar{A}, \quad \lim_{k \to \infty} E[b(X_k)] = 0
\]  

(16)

For mixing time \( \tau_\epsilon \) of \( \{X_k\} \) so that for all \( i \) and \( k \geq \tau_\epsilon \)

\[
\|E[b(X_k)|X_0 = i]\| \leq \epsilon, \quad \|E[A(X_k)|X_0 = i] - \bar{A}\| \leq \epsilon,
\]  

(17)

there exists \( K \geq 1 \) so that \( \tau_\epsilon \leq K \log \frac{1}{\epsilon} \).

2. Assumption 2:

\[
b_{\text{max}} := \sup_{i \in S} \|b(i)\| < \infty, \quad A_{\text{max}} := \sup_{i \in S} \|A(i)\| \leq 1
\]  

(18)

3. Assumption 3: \( A \) is Hurwitz: all eigenvalues have strictly negative parts

One can check that TD algorithms satisfy assumptions 1-3.
Relevant Quantities

1. Fact: there exists a symmetric matrix $P > 0$ so that

$$\bar{A}^t P + P \bar{A}^t = -I$$  \hspace{1cm} (19)

$\gamma_{\text{max}} := \text{largest eigenvalue of } P; \gamma_{\text{min}} := \text{smallest eigenvalue of } P$

2. some universal constants

$$k_1 = 62 \gamma_{\text{max}} (1 + b_{\text{max}}), \quad k_2 = 55 \gamma_{\text{max}} (1 + b_{\text{max}})^3, \quad \tilde{k}_2 = 2 (k_2 + \gamma_{\text{max}} b_{\text{max}}^2)$$ \hspace{1cm} (20)
Theorem Statement

Theorem

For $\epsilon$ so that $\kappa_1 \epsilon \tau_\epsilon + \epsilon \gamma_{\text{max}} \leq 0.05$ and all $k \geq \tau_\epsilon$,

$$
\mathbb{E}[\|\Theta_k\|^2] \leq \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\epsilon}{\gamma_{\text{max}}}ight)^{k-\gamma} (1.5\|\Theta_0\| + 0.5b_{\text{max}})^2 + \frac{\tilde{\kappa}_2 \gamma_{\text{max}}}{0.9\gamma_{\text{min}}} \epsilon T
$$

(21)

1. this is a finite error bound compared to the convergence result from Tsitsiklis and Van Roy 1997
2. if $k \geq \tau_\epsilon + O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)$, then $\mathbb{E}\|\Theta_k\|^2 = O(\epsilon \tau_\epsilon)$.
3. step size $\epsilon$ is fixed. Not difficult to extend analysis to algorithms with diminishing step sizes.
Theorem: Motivation

A standard way to study (1) is to consider

$$\mathbb{E}[W(\Theta_{k+1}) - W(\Theta_k) | H_k]$$

(22)

where $H_k$ is some appropriate history.

Two questions:

1. what is a suitable Lyapunov function $W$?
2. how to decide $H_k$?

To answer the first question, we rely on intuitions from

1. Stein’s Method
2. Stability (equilibrium) of the associated ODE
Stein’s Method: Taylor Expansion of Operator

Stochastic Recursion:

\[ \Theta_{k+1} = \Theta_k + \epsilon(A(X_k)\Theta_k + b(X_k)) \] \hspace{1cm} (23)

1. think about the problem in steady state + i.i.d. samples
2. for any proper function \( H \),
   \[ \mathbb{E}[H(\Theta_{k+1}) - H(\Theta_k)] = 0 \] \hspace{1cm} (24)
3. Taylor expansion:
   \[ \mathbb{E} \left[ \nabla^t H(\Theta_k)(\Theta_{k+1} - \Theta_k) + \frac{1}{2} (\Theta_{k+1} - \Theta_k)^t \nabla^2 H(\tilde{\Theta})(\Theta_{k+1} - \Theta_k) \right] = 0 \] \hspace{1cm} (25)

for appropriate \( \tilde{\Theta} \)
Stein’s Method: Poisson Equation

1. set up the Poisson equation:

   \[ \nabla^t W(\Theta_k) \mathbb{E}[\Theta_{k+1} - \Theta_k | \Theta_k] = -\|\Theta_k\|^2, \text{ for each } \Theta_k \]  

   (26)

2. Combining Poisson equation and Taylor expansion

   \[ \mathbb{E}[\|\Theta_k\|^2] = \mathbb{E} \left[ \frac{1}{2} (\Theta_{k+1} - \Theta_k)^t \nabla^2 W(\tilde{\Theta})(\Theta_{k+1} - \Theta_k) \right] \]  

   (27)

3. one can use Hessian bound to obtain bounds on \( \mathbb{E}[\|\Theta_k\|^2] \)

4. We focus on Poisson equation (26). By i.i.d. assumption,

   \[ \nabla^t W(\Theta_k) \overline{A} \Theta_k = -\|\Theta_k\|^2 \]  

   (28)
Stein’s Method: Intuition

1. Candidate solution to (28):

   \[ W(\Theta_k) = \Theta_k^t P \Theta_k \]  
   \[ (29) \]

   for \( P \) a symmetric positive definite matrix

2. Solve \( P \) so that

   \[ A^t P + PA^t = -I \]  
   \[ (30) \]

   The solution is unique due to the assumption that \( A \) is Hurwitz

3. Stein’s method (Poisson equation) removes the guesswork for a good Lyapunov function \( W \)
Stochastic Recursion:

$$\Theta_{k+1} = \Theta_k + \epsilon (A(X_k)\Theta_k + b(X_k))$$

(31)

1. the corresponding ODE:

$$\dot{\theta} = A\theta$$

(32)

2. Fact: $\Theta_k$ converges to the equilibrium point of ODE (32)

3. how one could derive bounds on $||\theta_t||^2$?
ODE: Same Lyapunov function

Consider

\[ W(\theta) = \theta^t P \theta \]  

(33)

1. consider the time derivative of \( W(\theta) \)

\[ \frac{dW}{dt} = \theta^t \left( \bar{A}^t P + P \bar{A}^t \right) \theta = -\| \theta \|^2 \]  

(34)

2. \( W(\theta) \leq \gamma_{\text{max}} \| \theta \|^2 \Rightarrow \frac{dW}{dt} \leq -\frac{1}{\gamma_{\text{max}}} W \)

3. Thus,

\[ \| \theta_t \|^2 \leq \frac{1}{\gamma_{\text{min}}} W(\theta_t) \leq \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} e^{-t/\gamma_{\text{max}}} \| \theta_0 \|^2 \]  

(35)

4. indicates that \( W \) is a correct choice of Lyapunov function
Two Methods, One Lyapunov Function and Similar Bounds

1. both Stein’s method and analysis of ODE point to the same Lyapunov function $W$

2. analysis of stochastic system is similar to ODE: drift of $W$ versus time derivative of $W$ along the trajectory of ODE

\[
E[\|\Theta_k\|^2] \leq \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\epsilon}{\gamma_{\text{max}}} \right)^{k-\gamma} (1.5\|\Theta_0\| + 0.5b_{\text{max}})^2 + \frac{\tilde{K}_2\gamma_{\text{max}}}{0.9\gamma_{\text{min}}} \epsilon T
\]  

\[
\sim \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} \left(1 - \frac{0.9\epsilon}{\gamma_{\text{max}}} \right)^{k-\gamma} \|\Theta_0\|^2
\]  

similar to $\frac{\gamma_{\text{max}}}{\gamma_{\text{min}}} e^{-t/\gamma_{\text{max}}} \|\theta_0\|^2$ for small $\epsilon$. 
How to Decide $H_k$?

1. Lyapunov function $W$ as a solution to Poisson equation: applying Stein’s method to steady state approximation
2. ODE is determined by the steady states of $A(X_k)$ and $b(X_k)$
3. given history $H_k$, for drift analysis of $W$ to be effective, we need to wait an initial transient period $\tau_\varepsilon$ for $A(X_k), b(X_k)$ close enough to steady states
4. $H_k := \Theta_{k-\tau}$
Proof of the Theorem

1. Use $W$ as Lyapunov function and obtain bound on the drift

$$
\mathbb{E}[W(\Theta_{k+1}) - W(\Theta_k)|\Theta_{k-\tau}] \leq -\frac{0.9\varepsilon}{\gamma_{\text{max}}} \mathbb{E}[W(\Theta_k)|\Theta_{k-\tau}] + k_2\varepsilon^2\tau\varepsilon
$$

(38)

2. Combine drift bound with

$$
\mathbb{E}\|\Theta_k\|^2 \leq \frac{1}{\gamma_{\text{min}}} \mathbb{E}[W(\Theta_k)]
$$

(39)

and various vector inequalities
References

R. Srikant, Lei Ying

*Finite-Time Error Bounds For Linear Stochastic Approximation and TD Learning.*

Mark Gluzman

*Multi-step learning and Value-based approximation methods.*